DATE OF EXAM **Solution**
SUBJECT NAME: Advanced Functional Analysis - FINAL OR MIDTERM Exam: Final-Semester 2016 - Semester ENTER I or II: M. Math IInd Yr

1. *Let X be a complex normed linear space. Suppose $L(X)$ the space of bounded linear operators with the usual norm. is a Banach space. Show that X is a Banach space.*

   Solution: Pick a Cauchy sequence in X, say $\{x_n\}_{n\geq 1}$. Given $x \in X$, define the mapping $f_x : X \to X$ by $f_x(y) = \phi(y)x$, where $\phi \in X^*$ is a fixed non zero functional. Let $y_0 \in X$ be such that $\phi(y_0) = 1$. Note that $||f_x|| \leq ||\phi||||x||$, so f_x is bounded and also it is easy to see that f_x is linear. Therefore, we have $\{f_{x_n}\}_{n\geq 1}$ is Cauchy sequence in $B(X)$ if $\{x_n\}$ is Cauchy in X. Let f be such that $f_{x_n} \to f$ then,
   $$f(y_0) = \lim_{n\to\infty} f_{x_n}(y_0) = \lim_{n\to\infty} \phi(y_0)x_n = \lim_{n\to\infty} x_n.$$
   □

2. *Let X be a normed linear space. Let $\{x_n\} \subset X$ be a sequence such that, the closed convex hull $K = \overline{CO(\{x_n\})}$ is compact set. Let $x_0 \in X$ and let $\mu = \sum \frac{1}{2^n}\delta(x_n)$. Suppose $x^*(x_0) = \int_K x^* d\mu$ for all $x^* \in X^*$. Show that $x_0 \in K$.*

   Solution: We have, $\int_K x^* d\mu = \sum\limits_n \frac{1}{2^n} \int x^* d\delta(x_n) = \sum\limits_n \frac{1}{2^n} x^*(x_n)$.

   $x^*(x_0) = \sum\limits_n \frac{1}{2^n} x^*(x_n) = x^*\left(\sum\limits_n \frac{1}{2^n} x_n\right)$.

   $x_0 = \sum\limits_n \frac{1}{2^n} x_n = \lim\limits_n \sum\limits_{k=1}^n \frac{1}{2^k} x_k = \lim\limits_n \left[\sum\limits_{k=1}^n \frac{1}{2^k} x_k + \frac{1}{2^n} x_n\right]$

   Therefore $x_0 \in K$.

   □

3. *Let X be a locally convex topological vector space. Let $F \subset X^*$ be a finite dimensional subspace. Show that there is a closed subspace $Y \subset X$ such that $Y^\perp = F$.*

   Solution: Since F is a finite dimensional subspace X^*. Let $\{f_1, f_2, ...f_n\}$ is a basis of F. We claim that $Y = \text{Ker}(f_1) \cap \text{Ker}(f_2) \cap ... \cap \text{Ker}(f_n)$. Observe that $f_i(Y) = 0$ for all $i = 1, 2, ..., n$, therefore $F \subseteq Y^\perp$. Using the Lemma 3.9 (page- 63 to 64) from the book Walter Rudin, Functional Analysis. We can see that if $f \in Y^\perp$ then f can be written as linear combination if f_i's, therefore f belongs to F, $Y^\perp \subseteq F$. Hence $Y^\perp = F$.

   □

4. *Let X, Y be a LCTVS spaces. Let $T : X \to Y$ be an isomorphism. Suppose X^* and Y^* are equipped with the weak*-topology. Show that $T^* : Y^* \to X^*$ is an isomorphism.*

   Solution: Let $T^* : Y^* \to X^*$ is defined by $T^*(y^*)(x) = y^*(T(x))$.
   We show that T^* is linear, let $y_1^*, y_2^* \in Y^*$, $T^*(y_1^* + y_2^*)(x) = T^*((y_1 + y_2)^*)(x) = (y_1 + y_2)^*(T(x)) = (y_1^* + y_2^*)(T(x)) = y_1^*(T(x)) + y_2^*(T(x)) = T^*(y_1^*)(x) + T^*(y_2^*)(x)$. Let c be a scalar and $y^* \in Y^*$, $T^*(cy^*)(x) = T^*((cy)^*)(x) = (cy)^*(T(x)) = cy^*(T(x)) = cT^*(cy^*)$.

We show that T^* is one to one and onto, $T^*(y_1^*)(x) = T^*(y_2^*)(x)$, $y_1^*(T(x)) = y_2^*(T(x))$ $\forall x$, thus $y_1^* = y_2^*$. Therefore T^* is one to one. For $x^* \in X^*$, take $y^* = x^* \circ T^{-1}$, then $T^*(y^*) = x^*$, hence T is onto.
We now show that T^* and $(T^*)^{-1}$ are continuous. $T^*(y^*) = y^* \circ T$, Since T is continuous therefore T^* is continuous. We know $(T^*)^{-1} = (T^{-1})^*$, the continuity of T^{-1} will imply that the continuity of $(T^*)^{-1}$. $\square$

5. *Give examples of two normed linear spaces, and a continuous linear map T between them such that T has closed range but the range of T^* is not closed.*

   Solution: Let c_{00} be space of all sequences which have only finitely many non zero elements with sup norm. Let $T : c_{00} \to c_{00}$ defined by $T((x_n)) = (\frac{x_n}{n})$. We can easily see that range of T is closed. We know that $c_{00}^* = l_1$. $T^* : l_1 \to l_1$ is $T^*((y_n)) = (\frac{y_n}{n})$. Let $Z^m = (1, \frac{1}{2}, ..., \frac{1}{m}, 0, 0, ...)$, $Z^m \in l_1$. $T^*(y^m) = (1, \frac{1}{2^2}, ..., \frac{1}{m^2}, 0, 0, ...)$. Taking limit $m \to \infty$ then $(1, \frac{1}{2^2}, ..., \frac{1}{m^2}, ...) \in l_1$. We have $T(1, \frac{1}{2}, ..., \frac{1}{m}, ...) = (1, \frac{1}{2^2}, ..., \frac{1}{m^2}, ...)$ but $(1, \frac{1}{2}, ..., \frac{1}{m}, ...)$ not belong to l_1. $\square$

6. *Show that the space of regular Borel probability measures on $[0, 1]$, equipped with the weak*-topology is a metrizable space.*

   Solution: Let $M([0, 1])$ be the space of regular Borel probability measures on $[0, 1]$. Since $[0, 1]$ is compact, we have $C[0, 1]$ is a separable Banach space. Denote the closed unit ball in $C([0, 1])^*$ by B. By Banach-Alaogu theorem B is weak*-compact, using Theorem 3.16 (page 70) in the book Walter Rudin, Functional Analysis B is metrizable. Let map $\wedge : M([0, 1]) \to C[0, 1]^*$ defined by $\wedge_\mu f : \int_{[0,1]} f d\mu$. $\wedge M([0, 1])$ is a closed subset of B, hence $\wedge M([0, 1])$ is a weak*-compact subset of $C([0, 1])^*$. Using Theorem 3.16 (page 70) in the book Walter Rudin, Functional Analysis $\wedge M([0, 1])$ is metrizable. Hence space of regular Borel probability measures on $[0, 1]$, equipped with the weak*-topology is a metrizable space. $\square$

7. *Let $D = \{z : |z| < 1\}$ be the open unit disk. Let $A(D)$ denote the space of analytic functions on D with family of semi-norms, $p_z(a) = |a(z)|$ for $z \in D$ and $a \in D$. Let $F = \{p \in A(D) : p \text{ is a polynomial of degree at most } n\}$. Show that $A(D) = F \bigoplus Y$ (direct sum) for some closed subspace $Y \subset A(D)$.* Let $P : A(D) \to Y$ be the canonical projection. Show that P is not a compact operator.

   Solution: Since $A(D)$ is locally convex topological vector space and dim $F = n+1 < \infty$. Using the Lemma 4.21(a), page-106 from the book Walter Rudin, Functional Analysis, F is complemented in $A(D)$. Therefore there exist a closed subspace $Y \subset A(D)$ such that $A(D) = F \bigoplus Y$.

   Suppose $P : A(D) \to Y$ is compact, we have range of P is closed. Using the Thorem 4.18(b) in the book Walter Rudin, Functional Analysis, thus dim range of P is finite. Which is a contradiction. Therefore P is not a operator. $\square$

8. *Let X be a normed linear space. State and prove the Banach-Alaogu theorem.*

   Solution: We can find the Statement and proof in the book Walter Rudin, Functional Analysis, Theorem 3.15, page-68.

   $\square$

9. *Let $T : l^2 \to l^2$ be defined by $T(\{\alpha_n\}) = \{\frac{\alpha_n}{n}\}$. Show that T is a compact operator.*

Solution: Define $T_N(\{\alpha_n\}) = (\alpha_1, \frac{\alpha_2}{2}, \frac{\alpha_3}{3}, ...\frac{\alpha_N}{N}, 0, 0, 0.....)$, since range of T_N is finite dimensional, therefore T_N is a compact operator. For $\{\alpha_n\} \in l^2$,

$$||T_N(\{\alpha_n\}) - T(\{\alpha_n\})||_2 = \left(\sum_{n>N} \left|\frac{\alpha_n}{n}\right|^2\right)^{\frac{1}{2}} \leq \left(\sup_{n>N} \left|\frac{1}{n}\right|\right) \left(\sum_{n>N} |\alpha_n|^2\right)^{\frac{1}{2}} \leq \left(\sup_{n>N} \left|\frac{1}{n}\right|\right) ||\{\alpha_n\}||_2$$

Therefore,

$$||T_N(\{\alpha_n\}) - T(\{\alpha_n\})||_2 \leq \sup_{n>N} \left|\frac{1}{n}\right|$$

Since $\frac{1}{n} \to 0$, it follows that $||T_N(\{\alpha_n\}) - T(\{\alpha_n\})||_2 \to 0$. Since set of compact operators form a closed subspace. Hence T is a compact operator.

$\square$

10. *Let $T \in L(X)$ be such that T^* maps extreme points of the unit ball of X^* to extreme points of the unit ball of X^*. Show that T is an extreme point of the unit ball of $L(X)$.*

Solution: Let x belongs to unit ball of X, We have

$$\begin{aligned} ||Tx|| &= ||y^*(Tx)||, \text{where } y^* \text{ belongs to the unit ball of } X^* \\ &= ||y^*(Tx)||, \text{where } y^* \text{ belongs to the extreme point of the unit ball of } X^* \\ &= ||T^*(y^*)(x)|| \leq 1. \end{aligned}$$

Therefore $||T|| \leq 1$.

Now we prove that T is an extreme point. Let $T = \lambda T_1 + (1-\lambda)T_2$, where $0 < \lambda < 1$, T_1 and T_2 belongs to the unit ball of $L(X)$. Let y^* belong to the extreme point of the unit ball of X^*, $y^*(T) = \lambda y^*(T_1) + (1-\lambda)y^*(T_2)$, $T^*(y^*) = \lambda T_1^*(y^*) + (1-\lambda)T_2^*(y^*)$. Since T^* maps extreme points of the unit ball of X^* to the extreme points of the unit ball of X^* we have,

$$T^*(y^*) = T_1^*(y^*) = T_2^*(y^*).$$

The above equation is true for every element in the unit ball of X^*, therefore for every element of X^*. Hence $y^*(T) = y^*(T_1) = y^*(T_2)$ for every $y^* \in X^*$. This implies that $T = T_1 = T_2$.

$\square$